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editation: The University of Alberta is a member of the Association of ersities and Colleges of Canada and the Association of Commonwealth ersities.

demic Terms: Effective September 1999, all periods of study at the University berta are represented by terms. There are four academic terms—Fall, Winter, g and Summer. The Fall and Winter Terms are 13 weeks long plus examination The Spring and Summer terms are six weeks long including examination time.

se Information: Course units are assigned for the purpose of calculating a hted (grade point) average. Normally, full courses consist of 6.0 units, include hours of lectures per week, and are offered over two terms. A full course presented on the transcript as two sections of the same course—section A 3. Section A of a full course is awarded a grade of IP. No weight is assigned s section. A final grade, the corresponding weight and grade point value are assigned to section B of the course. Half courses normally consist of 3.0 units and include three hours of instruction per week. Engineering courses have units assigned to them based on hours of instruction and may deviate from the norm.

Grade Point Average (GPA): The Grade Point Average (GPA) is a measure of a student's weighted average, obtained by dividing the total number of grade points earned by the total units of course weight attempted. Courses graded using grades of CR or NC, or from which students have withdrawn or have audited, are not calculated into the GPA. For more information, see §23.4(6) of the *University Calendar*.

Re-registration Policy: The university's re-registration policy, effective September 1988, does not normally allow students to repeat courses already passed or for which transfer credit has been granted. For more information, see §22.1.3 of the *University Calendar*.

Letter Grading System (effective September 2003)

se Grades Obtained by rgraduate Students

tor	Letter Grade	Grade Point Value
	A+	4.0
nt	A	4.0
	A-	3.7
	B+	3.3
	B	3.0
	B-	2.7
	C+	2.3
ctory	C	2.0
	C-	1.7
	D+	1.3
al Pass	D	1.0
	F	0.0

Course Grades Obtained by Graduate Students

Descriptor	Letter Grade	Grade Point Value
	A+	4.0
Excellent	A	4.0
	A-	3.7
Good	8+	3.3
	B	3.0
Satisfactory	B-	2.7
	C+	2.3
Failure	C	2.0
	C-	1.7
	D+	1.3
	D	1.0
	F	0.0

The 9-Point Grading System (before September 2003)

Course Grades Obtained by Undergraduate Students

	9	Outstanding
	8	Excellent
	7	Very Good
	6	Good
	5	Satisfactory
Pass	4	Minimally acceptable
Fail	3	Unsatisfactory
	2	
	1	

Course Grades Obtained by Graduate Students

	9	Excellent
	8	
	7	Good
Pass	6	Satisfactory
Fail	5	Unsatisfactory
	4	
	3	
	2	
	1	

Note: Effective September 1986, a grade of 3 is a failing grade for undergraduate students. Before September 1986, credit might be conceded by a Faculty for a grade of 3 in an individual course if warranted by a student's overall record.

Effective July 1989, a grade of 5 is a failing grade for graduate students. Before July 1989, credit might be conceded by the Faculty of Graduate Studies and Research for a grade of 4 or 5 in an individual course if warranted by a student's overall record.

er Final Grades

- aegrotat standing
- registered as an auditor
- registered as an auditor and withdrew
- completed requirements; no grade point value assigned
- exempt
- incomplete
- course in progress
- withdrew from or failed course in progress
- failure, no grade point value assigned
- withdrew with permission

arks

- grade includes a mark of '0' for final examination missed, or for term work missed, or both
- grade includes a mark of '0' for final examination missed, or for term work missed, or both, and deferred final examination granted
- credit withheld
- reexamination granted
- failure (assigned to failing Graduate Student grades only)
- failure, grade includes a mark of '0' for final examination missed, or for term work missed, or both (assigned to failing Graduate Student grades only)
- failure, grade includes a mark of '0' for final examination missed, or for term work missed, or both, and deferred final examination granted (assigned to failing Graduate Student grades only)
- disciplinary sanction for serious instances of inappropriate academic behavior
- failure, disciplinary sanction for serious instances of inappropriate academic behavior (assigned in the same circumstances as the remark of 8 but to failing Graduate Student grades only)

Other Final Grades

AB	absent from final examination
ABF	absent from final examination, failed, results in a grade of 1
AE	aegrotat standing
AU	registered as an auditor
AW	registered as an auditor and withdrew
CR	completed requirements; no numeric grade assigned
DB	debarred from final exam
DW	did not write
EX	exempt
F	failure; no numeric grade assigned
H	pass with honors standing
IN	incomplete
INF	incomplete, failed
IP	course in progress
IP*	withdrew from or failed course in progress
RC	registration cancelled
W	withdrew with permission
WF	withdrew failing, results in a grade of 1

Remarks

A	indicates missed final examination, term work missed, or both
B	grade includes '0' for missed term work and final examination
C	credit conceded
D	granted a deferred final examination
F	failure
FI	failed, inappropriate academic behavior
N	credit withheld
R	granted a reexamination
S	failure, supplemental granted
X	student failed to complete a significant part of term work

AN OPTIMAL PROPERTY OF THE POINCARÉ MODEL OF HYPERBOLIC GEOMETRY

by

Anton Alexander Cseuz

April, 1957.

FACULTY OF ARTS AND SCIENCE

UNIVERSITY OF ALBERTA

ABSTRACT

It is proved in this paper that the Poincaré model of hyperbolic geometry has the following optimal property:it is the conformal representation of the hyperbolic plane on the Euclidean plane which is the least distorted in a certain sense.

Some of the properties of the Poincaré model are reviewed.

In the end the numerical value of the distortion in the sense defined here is computed and a Mean Value Theorem deduced.

AN OPTIMAL PROPERTY OF THE POINCARÉ MODEL OF HYPERBOLIC GEOMETRY

by

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Under the direction of

Dr. H. Helfenstein

Department of Mathematics

University of Alberta

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CONTENTS

INTRODUCTION

In this paper we show an interesting property of the Poincaré model of hyperbolic geometry:namely,that it is the conformal representation of the hyperbolic plane on the Euclidean plane which is the least distorted in a certain sense.Distortion is defined here as the mean quadratic deviation of the logarithm of the scale (the ratio of corresponding line elements in the mapping) from a certain constant,taken over the whole hyperbolic plane.

We proceed as follows:we propose to find the conformal mapping of the hyperbolic plane onto the Euclidean plane, for which the distortion in the above sense becomes least.

The Poincaré model forms the logical starting point in our method(as will be seen).We give a discussion of a more general nature of the Poincaré model,before starting on the main problem in the second chapter.

CHAPTER I

(1). The Poincaré Model - a Conformal Model of Hyperbolic Geometry

The Poincaré model of hyperbolic geometry is a conformal mapping of the hyperbolic plane onto the unit circle in the Euclidean plane. If we denote the hyperbolic line element by ds_H and the corresponding line element in the Euclidean plane as ds_E , the mapping is defined by the formula

(1.1) $$ds_H = \frac{r_0}{1-x^2-y^2}\, ds_E \quad , \quad x^2+y^2<1 \,, \quad r_0>0 \,,$$

or in complex notation,

(2.1) $$ds_H = \frac{r_0}{1-z\bar{z}}\,|dz| \quad , \quad |z|<1 \quad .$$

It is easily checked that the surface defined by the line element ds_H is a hyperbolic plane, by calculating its Gaussian curvature.

If the square of the line element of a surface is given as

(3.1) $$(ds)^2 = E(du)^2 + G(dv)^2 \quad ,$$

the Gaussian curvature of the surface is given as:

$$K = -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right] .$$

Applying this formula to

$$(4.1) \qquad (ds)^2 = \frac{(r_o)^2}{(1-x^2-y^2)}\left(dx^2 + dy^2\right) ,$$

we get

$$K = -\frac{4}{(r_o)^2} .$$

Hence it is seen that the surface defined by ds_H is a hyperbolic plane;it can be isometrically mapped on a pseudosphere with pseudoradius $\frac{2}{r_o}$.

It is also easily verified that the mapping (1.1) is conformal.

On a surface defined by the line element in (3.1)

the cosine of the angle θ between any two line elements produced by independent increments du,dv and δu, δv is given by

$$\cos\theta = \frac{E\,du\,\delta u + G\,dv\,\delta v}{\sqrt{E(du)^2+G(dv)^2}\,\sqrt{E(\delta u)^2+G(\delta v)^2}} \quad .$$

Applying this to (4.1),we get

$$\cos\theta = \frac{dx\,\delta x + dy\,\delta y}{\sqrt{dx^2+dy^2}\,\sqrt{\delta x^2+\delta y^2}} \quad ,$$

which is precisely the formula for the angle between the line elements produced by the same increments at (x,y) in the Euclidean plane,for there,

$$(ds)^2 = (dx)^2 + (dy)^2 \quad .$$

Hence corresponding angles are equal.

(2) Images of Hyperbolic Straight Lines in Poincaré's Model.

We wish to determine what curves in the Poincaré model correspond to the geodesic lines on the hyperbolic plane, the hyperbolic straight lines.

Making a transformation of curvilinear coordinates in (4.1):

$$x = r\cos\theta \quad ,$$

$$y = r\sin\theta \quad ,$$

$$(dx)^2 + (dy)^2 = (dr)^2 + r^2(d\theta)^2 \quad ,$$

the square of line element becomes, with $(r_0)^2 = A$:

$$(1.2) \qquad (ds)^2 = \frac{A}{(1-r^2)^2}(dr)^2 + \frac{Ar^2}{(1-r^2)^2}(d\theta)^2 \quad .$$

For a line element of the form (3.1), the Christoffel symbols of the second kind $\left\{ {i \atop j\,k} \right\}$ constructed from E and G are obtained without difficulty using the set of formulas:

$$\left\{ {1 \atop 1\,1} \right\} = \frac{1}{2E}\frac{\partial E}{\partial u} \quad , \qquad \left\{ {2 \atop 2\,2} \right\} = \frac{1}{2G}\frac{\partial G}{\partial v} \quad ,$$

$$\left\{ {1 \atop 1\,2} \right\} = \frac{1}{2E}\frac{\partial E}{\partial v} \quad , \qquad \left\{ {2 \atop 2\,1} \right\} = \frac{1}{2G}\frac{\partial G}{\partial u} \quad ,$$

$$\left\{ {1 \atop 2\ 2} \right\} = -\frac{1}{2E}\frac{\partial G}{\partial u} \quad , \quad \left\{ {2 \atop 1\ 1} \right\} = -\frac{1}{2G}\frac{\partial E}{\partial v} \quad .$$

Applying these to (1.2) with

$u = r$

$v = \theta$ and hence

$$E = \frac{A}{(1 - r^2)^2} \quad , \qquad G = \frac{Ar^2}{(1 - r^2)^2} \quad ,$$

we obtain for the nonvanishing Christoffel symbols

$$\left\{ {1 \atop 1\ 1} \right\} = \frac{2r}{(1 - r^2)}$$

$$\left\{ {2 \atop 2\ 1} \right\} = \frac{1 + r^2}{r(1 - r^2)}$$

$$\left\{ {1 \atop 2\ 2} \right\} = -\frac{r(1 + r^2)}{(1 - r^2)} \quad .$$

The equation of geodesics on the surface with line element of the form (3.1) is

$$\frac{d^2v}{du^2} = \left\{ {1 \atop 2\ 2} \right\}\left(\frac{dv}{du}\right)^3 + \left(2\left\{ {1 \atop 1\ 2} \right\} - \left\{ {2 \atop 2\ 2} \right\}\right)\left(\frac{dv}{du}\right)^2$$

$$+ \left(\left\{ {1 \atop 1\ 1} \right\} - 2\left\{ {2 \atop 1\ 2} \right\}\right)\frac{dv}{du} - \left\{ {2 \atop 1\ 1} \right\} \quad .$$

In our case this becomes

$$\frac{d^2\theta}{dr^2} = -\frac{r(1+r^2)}{1-r^2}\left(\frac{d\theta}{dr}\right)^3 - \frac{2}{r(1-r^2)}\left(\frac{d\theta}{dr}\right) .$$

This is also the equation of the images of the geodesics in the Euclidean plane.

Since this is a differential equation with the coefficients functions of r only, we put

$$\frac{d\theta}{dr} = p$$

and obtain

$$p^{-3}\frac{dp}{dr} + \frac{2}{r(1-r^2)}p^{-2} = -\frac{r(1+r^2)}{1-r^2},$$

which after the substitution

$$y = p^{-2}$$

becomes

$$\frac{dy}{dr} - \frac{4}{r(1-r^2)}y = \frac{2r(1+r^2)}{1-r^2} .$$

This is a linear differential equation of first degree. Carrying out the quadrature and substituting back, we obtain

$$\frac{d\theta}{dr} = i\,\frac{1 - r^2}{r\sqrt{1 + r^4}} \tag{2.2}$$

$$\theta = i\int \frac{1 - r^2}{r\sqrt{1 + r^4}}\,dr \quad .$$

This is a hyperbolic integral .Putting

$$r^2 = v \quad ,$$

we get

$$i\theta = \frac{1}{2}\csc^{-1}h\ v + \frac{1}{2}\sin^{-1}h\ v + C \ ,$$

or

$$i\theta = \frac{1}{2}\csc^{-1}h\ r^2 + \frac{1}{2}\sin^{-1}h\ r^2 + C \ , \text{ C is real.}$$

The solution of

$$i\theta - C = \frac{1}{2}\left(\frac{1 + (\sin^{-1}h\ r^2)^2}{\sin^{-1}h\ r^2}\right)$$

is

$$r^2 = + i \sin\left[(\theta + iC) \pm \sqrt{(\theta + iC)^2 + 1}\right] \ .$$

This is the equation of circles. These circles intersect the unit circle at right angles. For, using (2.2),

when $r = 1$,

$$\frac{d\theta}{dr} = 0 \quad .$$

On the other hand, on the unit circle $z = e^{i\theta}$,

$$\frac{dr}{d\theta} = 0 \quad \text{and hence}$$

$$\frac{d\theta}{dr} = \infty \quad .$$

Hence at the point of intersection the circular arcs are orthogonal to the unit circle.

The images of the hyperbolic straight lines in the Poincaré model of hyperbolic geometry are the circular arcs which intersect the unit circle at right angles.

CHAPTER II

(3) Statement of Problem and Notation.

Since the map (1.1) is conformal, it follows that all conformal mappimgs of this hyperbolic plane onto the Euclidean plane are of the form

$$f(z) = u + iv = w \quad ,$$

where $f(z)$ is regular for $|z| < 1, f'(z) \neq 0$ and u and v are real functions, i.e. they are the real and imaginary parts respectively of $f(z) = w$ in the complex w plane. In order to fix $f(z)$ wrt. rotations, translations and similarities, we arbitrarily set the values of $f(0)$, $\arg f'(0)$, and $|f'(0)|$.

Now let $f(z) = w$ be a function mapping the hyperbolic plane conformally onto the Euclidean plane, with the side conditions

$$(1.3) \quad \begin{cases} f(0) = 0 \\ \arg f'(0) = 0 \\ |f'(0)| = \mu r_o \quad , \quad \mu r_o > 0. \end{cases}$$

The scale in the point (x,y) of the conformal map determined by this $f(z)$ is given by

$$m(z) = \frac{|dw|}{ds_H} \quad .$$

$m(z)$ is a real function.

Since

$$|dw| = |f'(z)| \; |dz| \quad ,$$

by (2.1) we have

$$(2.3) \qquad m(z) = |f'(z)| \; \frac{|dz|}{ds_H} = |f'(z)| \; \frac{1 - z\bar{z}}{r_0} \quad .$$

Now

$$|f'(0)| = m(0) \; r_0 \quad .$$

Hence $m(0)$ is arbitrary by the last side condition (1.3), i.e.

$$m(0) = \mu \quad .$$

Our problem can now be stated as follows:

To find an $f(z)$ and a constant C such that the logarithm of the corresponding scale

$$m(z) = M(x,y)$$

satisfies the condition that the functional

$$(3.3) \quad I = \iint_U \left[\log M(x,y) - C\right]^2 dx \, dy \quad ,$$

where U is the unit circle in the Euclidean plane, becomes a minimum, with $f(z)$ subject to the side conditions (1.3).

Hence this is an extremal problem.

From (2.3) we observe that $\log M(x,y)$ is a real function, as both $|f'(z)|$ and $(1 - z\bar{z})$ are positive in $|z| < 1$.

Using the notation

$$\log |f'(z)| \equiv \Psi(x,y) \equiv \psi(z)$$

$$(4.3) \qquad \log \left(\frac{1 - z\bar{z}}{r_0} \right) \equiv \Phi(x,y) \equiv \varphi(z) \quad ,$$

(3.3) becomes

$$I = \iint_U (\Psi + \Phi - c)^2 dx\, dy \quad .$$

We observe that as $f'(z)$ is regular and $\neq 0$ in U,

$$\log f'(z) = \log |f'(z)| + i \arg f'(z)$$

is also regular, and hence $\log |f'(z)| \equiv \Psi(x,y)$ is a harmonic function, i.e. it satisfies Laplace's equation.

Rewriting the last side condition (1.3) in this notation, we have

$$\psi(0) = \log(\mu r_o) \quad .$$

(4) Calculation of "Best" Mapping Function f(z).

We put

$$z = re^{i\theta}$$

$$\varphi(z) = \varphi(re^{i\theta})$$

$$\psi(z) = \psi(re^{i\theta}) \quad .$$

φ and ψ being once differentiable real periodic functions of the real variable θ with period 2π, we form their Fourier expansions :

$$\varphi = \sum_{n=-\infty}^{\infty} a_n(r)\, e^{in\theta}$$

$$\psi + \varphi - C = \sum_{n=-\infty}^{\infty} b_n(r)\, e^{in\theta} ,$$

Comparison with (4,3) shows :

(1.4) $$a_o(r) = \log\left(\frac{1 - r^2}{r_o}\right) , \quad a_n(r) = 0, n \neq 0.$$

We can draw a number of conclusions about the b_n. Since $\psi + \varphi - C$ is real, from the formula for the

Fourier coefficients

$$b_n = \frac{1}{2\pi}\int_0^{2\pi}(\psi + \varphi - C)e^{-in\theta}\,d\theta$$

we deduce that

(2.4) $\qquad b_{-n} = \bar{b}_n \quad .$

Since

$$\psi(re^{i\theta}) + \varphi(re^{i\theta}) - C = b_0(r) + \left[b_1(r)e^{i\theta} + \overline{b_1(r)}e^{-i\theta}\right] + \dots$$

is one valued at each point $z = re^{i\theta}$, for $r = 0$ it must be one valued for arbitrary θ, and hence

(3.4) $$\begin{cases} b_0(0) = \psi(0) - \log r_0 - C \quad , \\ b_n(0) = 0 \quad \text{if} \quad n \neq 0 \quad . \end{cases}$$

Combining the two Fourier expansions, we obtain

(4.4) $$\psi = \sum_{n=-\infty}^{\infty}\left[b_n(r) - a_n(r)\right] e^{in\theta} + C \quad .$$

Since ψ was assumed to be harmonic, it is a solution of

(5.4) $$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0 \quad .$$

We proceed to the integration of this equation under the conditions (1.4),(2.4) and (3.4) .We use of course the method of separation of variables.

We set

$$\psi = \sum_{n=-\infty}^{\infty} R_n(r)\Theta_n(\theta) + C \quad .$$

Applying (3.4) to each $R_n(r)\Theta_n(\theta)$ separately,we have the equation

$$\Theta_n \frac{d^2R_n}{dr^2} + \frac{\Theta_n}{r}\frac{dR_n}{dr} + \frac{R_n}{r^2}\frac{d^2\Theta_n}{d\theta^2} = 0 \quad .$$

From this we get the two equations

$$\frac{1}{\Theta_n}\frac{d^2\Theta_n}{d\theta^2} = -n^2 \quad ,$$

$$r^2\frac{d^2R_n}{dr^2} + r\frac{dR_n}{dr} - n^2 R_n = 0 \quad .$$

Solving them we obtain

$$\psi = \sum_{n=-\infty}^{\infty} \left(A_n r^n + B_n r^{-n}\right)\left(C_n e^{in\theta} + D_n e^{-in\theta} \right) + C \; .$$

By comparison with (4.4) we have the relations

$$C_n = 1$$

$$D_n = 0$$

$$A_n r^n + B_n r^{-n} = b_n(r) - a_n(r)$$

$$b_n(r) = a_n(r) + A_n r^n + B_n r^{-n} \quad .$$

By (3.4),

$$\text{if} \quad n > 0, \quad B_n = 0 \quad ,$$

$$\text{if} \quad n < 0, \quad A_n = 0 \quad .$$

By (2.4),

$$A_n = \overline{B_{-n}} \quad .$$

Hence we can write, with a slight change of notation for the zero term,

(6.4) $$\begin{cases} b_0(r) = \log\left(\dfrac{1 - r^2}{r_0}\right) + A_0 \quad , \quad b_n(r) = A_n r^n \quad , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 1, 2, \ldots \\ b_{-n}(r) = \overline{b_n(r)} \end{cases}$$ (2.4)

(6.4) and (2.4) are independent equations.

The A_n and hence ψ are now determined by the minimum postulate. Here we set

$$I = \int_0^1 \int_0^{2\pi} (\psi + \varphi - C)^2 d\theta \; r dr.$$

Since

$$\psi + \varphi - C = \sum_{n=-\infty}^{\infty} b_n(r) \, e^{in\theta} \quad ,$$

$$\int_0^{2\pi} (\psi + \varphi - C)^2 d\theta = 2\pi \sum_{n=-\infty}^{\infty} b_n(r) \, b_{-n}(r) = 2\pi \sum_{n=-\infty}^{\infty} |b_n(r)|^2$$

$$= 2\pi |b_0(r)|^2 + 4\pi \sum_{n=1}^{\infty} |b_n(r)|^2 \quad ,$$

for $b_{-n}(r) = \overline{b_n(r)}$ and hence also $|b_n(r)|^2 = |b_{-n}(r)|^2$ by (24

Hence

$$I = \int_0^1 \int_0^{2\pi} (\psi + \varphi - C)^2 d\theta \; r dr$$

$$= 2\pi \int_0^1 |b_0(r)|^2 \; r dr + 4\pi \sum_{n=1}^{\infty} \int_0^1 |b_n(r)|^2 \; r dr \quad .$$

The validity of the interchange of summation and integration will be seen later.

Now by (6.4) ,

$$I = 2\pi \int_0^1 \left| A_0 + \log\left(\frac{1 - r^2}{r_0}\right)\right|^2 rdr$$

$$+ 4\pi \sum_{n=1}^{\infty} \int_0^1 \left|A_n r^n\right|^2 rdr \quad .$$

Now

$$\left| A_0 + \log\left(\frac{1 - r^2}{r_0}\right)\right|^2$$

$$= \left[A_0 + \log\left(\frac{1 - r^2}{r_0}\right)\right]\left[\bar{A}_0 + \log\left(\frac{1 - r^2}{r_0}\right)\right]$$

$$= |A_0|^2 + \bar{A}_0 \log\left(\frac{1 - r^2}{r_0}\right) + A_0 \log\left(\frac{1 - r^2}{r_0}\right) + \log^2\left(\frac{1 - r^2}{r_0}\right),$$

hence

$$\int_0^1 \left| A_0 + \log\left(\frac{1 - r^2}{r_0}\right)\right|^2 rdr - \int_0^1 \log^2\left(\frac{1 - r^2}{r_0}\right) rdr$$

$$= \frac{|A_0|^2}{2} + \bar{A}_0 \int_0^1 \log\left(\frac{1 - r^2}{r_0}\right) rdr + A_0 \int_0^1 \log\left(\frac{1 - r^2}{r_0}\right) rdr$$

$$= \left|\frac{A_0}{\sqrt{2}} + \sqrt{2}\int_0^1 \log\left(\frac{1 - r^2}{r_0}\right) rdr\right|^2 - 2\left|\int_0^1 \log\left(\frac{1 - r^2}{r_0}\right) rdr\right|^2 ,$$

for, if P and Q are any two complex numbers,

$$|P + Q|^2 - Q^2 = |P|^2 + P\bar{Q} + \bar{P}Q \,, \quad \text{as}$$

$$|P + Q|^2 = (P + Q)(\bar{P} + \bar{Q}) = |P|^2 + |Q|^2 + P\bar{Q} + \bar{P}Q \,.$$

Hence, finally,

$$I = 2\pi\left|\frac{A_0}{\sqrt{2}} + \sqrt{2}\int_0^1 \log\left(\frac{1 - r^2}{r_0}\right) r dr\right|^2 -$$

$$4\pi\left|\int_0^1 \log\left(\frac{1 - r^2}{r_0}\right) r dr\right|^2 + 2\pi\int_0^1 \log^2\left(\frac{1 - r^2}{r_0}\right) r dr +$$

$$2\pi\sum_{n=1}^{\infty}\frac{|A_n|^2}{n+1} \,.$$

Hence, when I attains its minimum value, say I_{min}, we have the following values for the A_n :

$$(7.4) \qquad A_0 = -2\int_0^1 \log\left(\frac{1 - r^2}{r_0}\right) r dr = 1 + \log r_0 \,, \quad A_n = 0 \,, \quad n = 1, 2, \dots \,.$$

Hence

$$b_0(r) = \log\left(\frac{1 - r^2}{r_0}\right) + 1 + \log r_0 \,, \quad b_n(r) = 0, \quad n = 1, 2, \dots \,.$$

and the interchange of summation and integration is therefore seen to be justified.

The formula for I_{min} is

$$I_{min} = 2\pi \int_0^1 \log^2\left(\frac{1 - r^2}{r_0}\right) r\,dr - 4\left|\int_0^1 \log\left(\frac{1 - r^2}{r_0}\right) r\,dr\right|^2 .$$

To obtain C, by (3.4)

$$C = \psi(0) - \log r_0 - b_0(0) ,$$

and by (6.4), this becomes

$$C = \psi(0) - A_0 .$$

Using the initial condition $\psi(0) = \log(\mu r_0)$, we get

$$C = \log\mu - I .$$

Hence we see that the constant is also uniquely determined by our minimum postulate.

Substituting in the formula for ψ we finally obtain

(8.4) $$\psi(re^{i\theta}) = A_0 + C = \log(r_0\mu) .$$

Thus,

$$\log|f'(z)| = \log(r_0\mu)$$

$$|f'(z)| = r_0\mu .$$

Since the right hand side is devoid of x and y, we finally obtain our best mapping function by integrating under the side conditions (1.3):

$$f(z) = r_0 \mu z \quad .$$

Hence the mapping merely produces an elongation of distances, or in other words, the best conformal map in our sense is the Poincaré model.

CHAPTER III

(5) Value of I_{min}.

The value of I_{min} is

$$I_{min} = 2\int_0^1 \log^2\left(\frac{1-r^2}{r_0}\right) r\,dr \; - 4\pi \left[\int_0^1 \log\left(\frac{1-r^2}{r_0}\right) r\,dr\right]^2 .$$

We apply integration by parts. For the second integral on the right hand side: $\int \log v \, dv = v \log v - v + C$. We obtain the value $-\pi\,(1 + \log r_0)^2$ for this term.

For the other term, we have the expression $\int \log^2 v \, dv = v \log^2 v - 2v \log v + 2v + C$ and the value $\pi(\log^2 r_0 + 2 \log r_0 + 2)$. Hence

$$I_{min} = \pi .$$

For $r_0 \to \infty$ or $K = 0$, that is for hyperbolic geometry approaching Euclidean geometry, we would expect from the definition of I as total distortion that $I_m = 0$. Hence $I_{min}(K)$ is discontinuous at $K = 0$.

(6) Mean Value Theorem.

Theorem. $\psi(re^{i\theta}) - \dfrac{C}{4r}$ has the same mean value over the unit circle as $\varphi(re^{i\theta}) - \dfrac{C}{4r}$, but opposite in sign.

Proof. From (8.4),

$\psi(re^{i\theta}) = A_0 + C$, and we get by (7.4)

$$\frac{1}{2\pi}\int_0^{2\pi}\psi(re^{i\theta})\,d\theta = -2\int_0^1 \log\left(\frac{1-r^2}{r_0}\right) rdr + C$$

$$= -\frac{1}{\pi}\int_0^1\int_0^{2\pi}\varphi(re^{i\theta})\,rdr + C$$

Multiplying both sides by r, and integrating between $r=0$ and $r=1$, we get

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\psi(re^{i\theta})\,d\theta\,rdr = -\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\varphi(re^{i\theta})\,d\theta\,rdr + \frac{C}{2}\,.$$

Hence

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\psi(re^{i\theta})\,d\theta\,rdr - \frac{C}{4} = -\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\varphi(re^{i\theta})\,rdr\,d\theta + \frac{C}{4},$$

and the theorem follows.

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